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On Badly Approximable Functions*

T. W. GAMELIN

Department of Mathematics, University of California, Los Angeles

J. B. GARNETT

Department of Mathematics, University of California, Los Angeles

L. A. RUBEL

Department of Mathematics, University of Illinois at Urbana–Champaign, Illinois

AND

A. L. SHIELDS

Department of Mathematics, University of Michigan, Ann Arbor

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1. INTRODUCTION

Let D be a bounded domain in the complex plane with boundary Γ, and let $A(D)$ be the algebra of analytic functions on D which extend continuously to Γ. The distance from a function $\varphi \in C(\Gamma)$ to $A(D)$ is defined to be

$$d(\varphi, A(D)) = \inf\{\| \varphi - f \| : f \in A(D)\},$$

where the norm is the supremum norm over Γ. In this paper, we consider the problem of describing the functions $\varphi \in C(\Gamma)$ which satisfy

$$\| \varphi \| = d(\varphi, A(D)).$$

Such functions, excepting the function 0, will be called *badly approximable*. Thus a function is badly approximable if its best analytic approximant is 0.

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For φ a nonvanishing function on Γ, there is a unique integer m with the following property: there is a continuous nonvanishing function f on $\bar{D}$ such that for $z_0 \in D$, the function $\varphi f/(z - z_0)^m$ has a continuous logarithm on Γ. The integer m is called the *index of* φ, and denoted by $\mathrm{ind}(\varphi)$. If Γ consists of a finite number of simple closed disjoint Jordan curves, then $\mathrm{ind}(\varphi)$ is the usual winding number of φ around Γ.

Our aim in this paper is to prove simply and to extend to more general domains, the following theorem of Poreda [10].

POREDA'S THEOREM. *Suppose Γ consists of a simple closed Jordan curve. Then $\varphi \in C(\Gamma)$ is badly approximable if and only if φ has nonzero constant modulus and* $\mathrm{ind}(\varphi) < 0$.

Half of Poreda's theorem extends trivially to arbitrary domains, as follows.

THEOREM 1.1. *If $\varphi \in C(\Gamma)$ has nonzero constant modulus, and if* $\mathrm{ind}(\varphi) < 0$, *then φ is badly approximable.*

Proof. Suppose $|\varphi| = 1$, and φ is not badly approximable. It suffices to show that $\mathrm{ind}(\varphi) \geqslant 0$. For this, choose $g \in A(D)$ such that $\|\varphi - g\| < \|\varphi\| = 1$. Then $\|1 - \bar{\varphi}g\| < 1$, so that $\bar{\varphi}g$ is an exponential, and φ and g have the same index. Since the index of an analytic function is nonnegative, $\mathrm{ind}(\varphi) \geqslant 0$. Q.E.D.

In Section 2, we give an elementary proof of the remaining implication of Poreda's theorem.

A point $z \in \Gamma$ is an *$A(D)$-essential boundary point* of D if for each neighborhood of z there exists a function in $A(D)$ which does not extend analytically to that neighborhood. The $A(D)$-essential boundary points form a closed subset of Γ which includes the boundary of the complement of $\bar{D}$.

In Section 3, a simple duality argument is used to prove the following.

THEOREM 1.2. *Each badly approximable function in $C(\Gamma)$ has constant modulus on the set of $A(D)$-essential boundary points of D.*

Theorem 1.2 reduces questions about badly approximable functions to the unimodular case. It turns out (Section 6) that for certain domains D, there are unimodular functions in $C(\Gamma)$ with arbitrarily large winding numbers, which are still badly approximable. Our principal result is the following partial converse the Theorem 1.1.

THEOREM 1.3. *Suppose that Γ consists of $N + 1$ disjoint closed Jordan curves. If $\varphi \in C(\Gamma)$ is badly approximable, then φ has nonzero constant modulus, and*

$$\mathrm{ind}(\varphi) < N.$$

Theorems 1.1 and 1.3 include Poreda's theorem, which corresponds to the case $N = 0$. One proof of Theorem 1.3, using the dual extremal method, is given in Sections 4 and 5. A second proof, using Toeplitz operators, is given in Section 7. An example given in Section 6 shows that the range $0 \leqslant \operatorname{ind}(\varphi) < N$ is indeterminate. Finally, in Section 8 we extend the results to finite Riemann surfaces.

2. An Elementary Proof of Poreda's Theorem

It suffices to consider the case in which D is the open unit disc Δ. Let $\varphi \in C(\Gamma)$ satisfy $\| \varphi \| = 1$. In view of Theorem 1.1, it suffices to show that either of the conditions

$$\varphi \text{ is not unimodular,} \tag{2.1}$$

$$\varphi \text{ is unimodular and } \operatorname{ind}(\varphi) \geqslant 0, \tag{2.2}$$

implies that $d(\varphi, A(\Delta)) < 1$.

Suppose that (2.1) is valid. Choose $b < 1$ so near to 1 that the set $E = \{w \in \Gamma : b \leqslant |\varphi(w)| \leqslant 1\}$ is a proper subset of Γ. Then E is simply connected, so that $\arg(\varphi)$ has a continuous determination on E. Consequently there is a smooth function $v \in C_R(\Gamma)$ such that $| \arg(\varphi) - v | \leqslant \pi/4$ on E. The harmonic conjugate $^*v$ of v is then continuous on Γ [14], and $g = \exp(iv - {}^*v)$ belongs to $A(\Delta)$. The range of g/φ on E is contained in the sector $\{| \arg z | \leqslant \pi/4\}$, so that for $\delta > 0$ sufficiently small, the range of $\delta g/\varphi$ on E is contained in the open disc centered at 1 with radius 1. Hence

$$| \varphi - \delta g | \leqslant | \varphi || 1 - \delta g/\varphi | < 1$$

on E. If $\delta > 0$ is small, also $| \varphi - \delta g | < 1$ on $\Gamma \backslash E$, so that $\| \varphi - \delta g \| < 1$, and $d(\varphi, A(\Delta)) < 1$.

Next suppose that (2.2) is valid, and set $m = \operatorname{ind}(\varphi)$. Write $\varphi = z^m e^{iu}$, where $u \in C_R(\Gamma)$. Let $v \in C_R(\Gamma)$ be a smooth function which satisfies $\| u - v \| \leqslant \pi/4$. As before, set $g = \exp(iv - v^*) \in A(\Delta)$. Again the range of $\delta g e^{-iu}$ is contained in the open disc centered at 1 with radius 1, for $\delta > 0$ sufficiently small. Consequently

$$d(\varphi, A(\Delta)) \leqslant \| \varphi - \delta z^m g \| = \| 1 - \delta g e^{-iu} \| < 1.$$

This completes the proof.

3. Dual Extremal Measures

Let $A(D)^{\perp}$ denote the (finite regular Borel) measures on Γ which are orthogonal to $A(D)$. By the Hahn–Banach theorem, there is for each $\varphi \in C(\Gamma)$ a measure $\mu \in A(D)^{\perp}$ such that $\| \mu \| = 1$ and

$$d(\varphi, A(D)) = \int \varphi \, d\mu.$$

If φ is badly approximable, then the chain of inequalities $\| \varphi \| = d(\varphi, A(D)) = \int \varphi \, d\mu \leqslant \int | \varphi | \, d | \mu | \leqslant \| \varphi \|$ become all equalities. We conclude that

$$\varphi\mu \geqslant 0 \tag{3.1}$$

$$| \varphi | = \| \varphi \| \text{ on the closed support of } \mu. \tag{3.2}$$

Conversely, if there is a nonzero measure $\mu \in A(D)^{\perp}$ for which (3.1) and (3.2) are valid, then

$$d(\varphi, A(D) \geqslant \int \varphi \, d\mu \Big/ \int d \, | \mu | = \| \varphi \|,$$

so that φ is badly approximable.

Any nonzero measure $\mu \in A(D)^{\perp}$ satisfying (3.1) and (3.2) is called a *dual extremal measure* for φ. Then $\varphi \in C(\Gamma)$ is badly approximable if and only if there is a dual extremal measure for φ. Theorem 1.2 is now an immediate consequence of (3.2) and the following lemma.

Lemma 3.1. *If μ is a nonzero measure in $A(D)^{\perp}$, then the closed support of μ contains the $A(D)$-essential boundary points of D.*

Proof. We will use some facts about the Cauchy transform $\hat{\tau}$ of a measure τ on Γ, defined by

$$\hat{\tau}(z) = \int \frac{d\tau(\zeta)}{\zeta - z}.$$

The integral converges absolutely for almost all $(dx\,dy)$ complex numbers z, and $\hat{\tau}$ is analytic off the closed support of τ. If $\hat{\tau} = 0$ a.e. $(dx\,dy)$, then $\tau = 0$. Finally, if $\tau \in A(D)^{\perp}$, then $\hat{\tau} = 0$ a.e. $(dx\,dy)$ on the complement of D, so that τ is completely determined by the analytic function $\hat{\tau}$ on D [3, Lemma 1.1].

Now let μ be a nonzero measure in $A(D)^{\perp}$, and suppose $z_0 \in \Gamma$ does not lie in the closed support of μ. Choose $\delta > 0$ so that the disc $\Delta_0 = \{| z - z_0 | < \delta\}$ carries no mass for μ. Then $\hat{\mu}$ is analytic on Δ_0, and $\hat{\mu}$ is not identically zero on D. Hence $\hat{\mu}$ vanishes on no open subset of Δ_0, and $\Delta_0 \subset \bar{D}$.

Let $f \in A(D)$. For fixed $z \in D$, the function $[f(z) - f(\zeta)]/(z - \zeta)$, regarded as a function of ζ, belongs to $A(D)$. Hence

$$\int \frac{f(z) - f(\zeta)}{z - \zeta}\, d\mu(\zeta) = 0, \qquad z \in D.$$

Solving for $f(z)$, we obtain

$$f(z) = \frac{1}{\hat{\mu}(z)} \int \frac{f(\zeta)\, d\mu(\zeta)}{\zeta - z}, \qquad z \in D.$$

This formula shows that f extends meromorphically to Δ_0. The meromorphic extension must coincide with the continuous extension of f from D to $\bar{D}$, so that f is analytic on Δ_0. Consequently z_0 is not an $A(D)$-essential boundary point of D. That proves the lemma.

On the basis of Lemma 3.1 it is easy to see that the set of $A(D)$-essential boundary points of D coincides with the Shilov boundary of $A(D)$.

4. Some Preparatory Lemmas

For $p > 0$, the space $H^p(V)$ associated with a domain V consists of the analytic functions f on V such that $|f|^p$ has a harmonic majorant. If J is an analytic arc which forms a relatively open subset of ∂V, then the nontangential boundary values of such an f exist almost everywhere with respect to the arc length measure on J. The boundary value function will also be denoted by f.

As usual, the open unit disc will be denoted by Δ. A theorem of Helson and Sarason [6] and Neuwirth and Newman [9] asserts that if $f \in H^{1/2}(\Delta)$ has positive radial boundary values a.e.$(d\theta)$ on $\partial\Delta$, then f is constant. The main idea of their proofs also serves to establish the following local version, which is due to Koosis [7].

Lemma 4.1. *Let $f \in H^{1/2}(\Delta)$, and let J be an open arc on $\partial\Delta$. If the radial boundary values of f are positive a.e. $(d\theta)$ on J, then f extends analytically across J.*

Since the proof is brief, we include it. Write $f = BF^2$, where B is a Blaschke product and $F \in H^2(\Delta)$. The condition that $f(z) \geqslant 0$ on J becomes the condition $B(z)\,F(z) = \bar{F}(1/\bar{z})$ on J. The result of the lemma now follows from the H^1 version of Morera's theorem (cf. [11]), which shows that if $g \in H^2(\Delta)$ agrees on an arc J of $\partial\Delta$ with a function $G \in H^1(\{|z| > 1\})$, then g extends analytically across J.

A conformally invariant statement of Lemma 4.1 is as follows.

LEMMA 4.2. *Let V be a domain, and let J be an analytic arc which forms an open subset of ∂V. Suppose $f \in H^{1/2}(V)$ satisfies $fdz \geqslant 0$ along J. Then f extends analytically across J.*

In the following lemma, we do not know whether the ϵ can be taken to be 0.

LEMMA 4.3. *Let V be a domain, and let J be an analytic arc which forms an open subset of ∂V. Let $\epsilon > 0$, and let $f \in H^{\epsilon+1/2}(V)$. If there is a continuous unimodular function φ on J such that $\varphi f\, dz \geqslant 0$ along J, then f is of class H^p, for all $p < \infty$, near each compact subarc of J.*

Proof. The problem is local, so that we can assume that $V = \Delta$. Let I be a relatively compact subarc of J, and let $u \in C_R(\partial\Delta)$ satisfy $\varphi = e^{iu}$ near I. By [14, Chap. VII, Theorem 2.11(ii)], $\exp(iu - {}^*u)$ is of class H^p for all $p < \infty$. Hence $g = \exp(iu - u^*) f \in H^{1/2}(\Delta)$. Furthermore, $gdz \geqslant 0$ along I. By Lemma 4.2, g extends analytically across I. Since $\exp(-iu + {}^*u)$ also is of class H^p for all finite p, f is of class H^p for all $p < \infty$, near compact subsets of I.

The following lemma is a standard variant of the argument principle [8, Chap. III, Sec. 10].

LEMMA 4.4. *Suppose that the boundary Γ of D consists of $N + 1$ simple closed analytic Jordan curves. Suppose f is meromorphic on a neighborhood of $\bar{D}$, and* $\arg(fdz)$ *is constant on each component of Γ. Then the difference of the number of zeros of f and the number of poles of f on $\bar{D}$ is $N - 1$. (Here the zeros or poles of f on Γ are counted according to half their multiplicity.)*

5. PROOF OF THEOREM 1.3

To prove Theorem 1.3, we can and will assume that the boundary Γ of D consists of $N + 1$ simple closed analytic Jordan curves. In this case, the measures $\mu \in A(D)^{\perp}$ are precisely the measures of the form $\mu = fdz$, when $f \in H^1(D)$ (cf. [11]). A dual extremal measure will be referred to as a *dual extremal differential.* Let φ be a unimodular function in $C(\Gamma)$. Denote by Γ_0 the "outside" component of Γ, and by $\Gamma_1, \ldots, \Gamma_N$ the "inside" components. Let z_j be any fixed point inside Γ_j, $1 \leqslant j \leqslant N$, and let $z_0 \in D$. For appropriate integers $m_1, \ldots, m_N$, we can express

$$\varphi(z) = [(z - z_0)/| z - z_0 |]^{\mathrm{ind}(\varphi)}\, e^{iv} g(z)/| g(z)| \,. \tag{5.1}$$

where $v \in C_R(\Gamma)$, and

$$g(z) = (z - z_1)^{m_1} \cdots (z - z_N)^{m_N}$$

is an invertible function in $A(D)$. Define $u_j \in C_R(\Gamma)$ to be 1 on Γ_j and 0 on $\Gamma \backslash \Gamma_j$, $1 \leqslant j \leqslant N$. Then there are constants $c_1, \ldots, c_N$ such that

$$u = v - \sum c_j u_j$$

has a single-valued harmonic conjugate function $^*u$ on D [8, Chap. I, Sect. 10]. Define $\varphi_0 = \exp(i \sum c_j u_j)$, so that

$$\begin{aligned} \varphi_0 &= 1 \qquad \text{on } \Gamma_0, \\ &= \exp(ic_j) \qquad \text{on } \Gamma_j,\ 1 \leqslant j \leqslant N. \end{aligned} \tag{5.2}$$

The formula (5.1) becomes

$$\varphi = \varphi_0[(z - z_0)/|z - z_0|]^{\operatorname{ind}(\varphi)} e^{iu} g(z)/|g(z)|. \tag{5.3}$$

Now suppose that φ is badly approximable, and that $f dz \in A(D)^\perp$ is a dual extremal differential. By Lemma 4.3, f is of class H^p on D, for all $p < \infty$. Consequently the function

$$G = fg \exp(iu - {}^*u) \tag{5.4}$$

belongs to H^p for all $p < \infty$. The relation $\varphi f dz \geqslant 0$ becomes

$$\varphi_0 (z - z_0)^{\operatorname{ind}(\varphi)} G dz \geqslant 0 \text{ along } \Gamma. \tag{5.5}$$

By Lemma 4.2, G extends analytically across Γ. Furthermore, the meromorphic differential $(z - z_0)^{\operatorname{ind}(\varphi)} G dz$ has constant argument along each component of Γ. From Lemma 4.4, we conclude that G has $N - 1 - \operatorname{ind}(\varphi)$ zeros on $\bar{D}$, where the zeros of G on Γ are counted according to half their multiplicity. Setting $F = G/g$, we obtain the following.

THEOREM 5.1. *Let φ be a unimodular function in $C(\Gamma)$ which is badly approximable, and let $f dz$ be a nonzero dual extremal differential for φ. Then there are $u \in C_R(\Gamma)$ and an analytic function F on $\bar{D}$, such that u has a single-valued harmonic conjugate $^*u$, and*

$$f(z) = F(z) \exp(-iu + {}^*u).$$

Furthermore, F has $N - 1 - \operatorname{ind}(\varphi)$ zeros on $\bar{D}$, where the zeros of F on Γ are counted according to half their multiplicity.

Theorem 1.3 is an immediate consequence of Theorem 5.1. Indeed, since the number of zeros of F cannot be negative, we obtain from Theorem 5.1 the estimate

$$\operatorname{ind}(\varphi) \leqslant N - 1$$

whenever φ is badly approximable.

Now return to the formula (5.3), and suppose that $\text{ind}(\varphi) = 0$. If φ is badly approximable, with dual extremal differential $f dz$, then (5.4) and (5.5) show that φ_0 is also badly approximable. Conversely, if φ_0 is badly approximable. with dual extremal differential G, then the H^1 function f defined by (5.4) satisfies $\varphi f dz \geqslant 0$, so that φ is badly approximable. We conclude that φ and φ_0 are simultaneously badly approximable or not, when $\text{ind}(\varphi) = 0$.

Unfortunately we do not know which locally constant functions φ_0 are badly approximable. At first we suspected that a locally constant unimodular function φ_0 is not badly approximable if and only if its range lies on a subarc of $\partial\Delta$ of length less than π. An example given in the next section shows that this guess fails. The following trivial observation, valid for arbitrary domains, is sufficient to lead to complete information in the case of an annulus.

LEMMA 5.2. *Let φ_0 be a continuous unimodular function on ∂D which assumes only two values. Then φ_0 is badly approximable if and only if the two values are diametrically opposite each other.*

Proof. If the values of φ_0 are not diametrically opposed, then their average g is a constant function which satisfied $\| \varphi_0 - g \| < 1$, so that φ_0 is not badly approximable. On the other hand, if the values are diametrically opposed, then no $h \in A(D)$ can satisfy $\| \varphi_0 - h \| < 1$, or else there would be a line passing through 0 and separating the range of h on Γ_0 from the range of h on Γ_1, an absurdity.

THEOREM 5.3. *Fix $0 < r < 1$, and let D be the annulus $\{r < |z| < 1\}$. Let φ be a unimodular function in $C(\Gamma)$. Then φ is badly approximable if and only if either* $\text{ind}(\varphi) < 0$, *or* $\text{ind}(\varphi) = 0$ *and*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \arg(\varphi(e^{i\theta}))\, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \arg(\varphi(re^{i\theta}))\, d\theta \equiv \pi (\text{mod } 2\pi) \qquad (5.6)$$

[*Here we integrate continuous determinations of* $\arg(\varphi)$ *on the respective intervals of integration.*]

Proof. In view of Theorems 1.1 and 1.3, it suffices to consider the case $\text{ind}(\varphi) = 0$. With $z_1 = 0$, the formula (5.2) then becomes

$$\begin{aligned} \varphi(z) &= z^m e^{iu}, & |z| &= 1 \\ &= (z^m / r^m)\, e^{iu} e^{ic}, & |z| &= r. \end{aligned}$$

Moreover, $\varphi_0 = 1$ on Γ_0, and $\varphi_0 = e^{ic}$ on Γ_1. By Lemma 5.2 and the remarks preceding that lemma, φ is badly approximable if and only if φ_0 is, and this occurs if and only if $c \equiv \pi$ (mod 2π). Now $u + i{}^*u$ is analytic, so that $\int_{-\pi}^{\pi} u(e^{i\theta})\, d\theta = \int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta$, and the left-hand side of (5.6) is computed to be $-c(\text{mod } 2\pi)$. This completes the proof.

6. Some Examples

First we show that the estimate of Theorem 3.1 is sharp, for any D. It suffices to consider the case in which the $N + 1$ Jordan curves which form the boundary Γ of D are analytic. In this case, set

$$\varphi = d\bar{z}/ds,$$

when ds is the arc length measure on Γ. Then φ is continuous and unimodular on Γ. Since the argument of dz/ds increases by 2π around the "outside" contour of Γ, and it decreases by 2π around each of the N "inside" contours of Γ, the index of dz/ds is $1 - N$, and

$$\operatorname{ind}(\varphi) = N - 1.$$

Since $\varphi\, dz \geqslant 0$, dz is a dual extremal differential for φ, and φ is badly approximable.

More generally, for any integer $k \geqslant 0$, there is a badly approximable function on ∂D with index $N - 1 - k$. Indeed, for fixed $z_0 \in D$, the function

$$\frac{|z - z_0|^k}{(z - z_0)^k} \frac{d\bar{z}}{ds}$$

has index $N - 1 - k$. Since it has the dual extremal differential $(z - z_0)^k\, dz$, it is badly approximable.

The remaining examples depend on the following lemma.

Lemma 6.1. *Suppose that Γ consists of $N + 1$ disjoint circles $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, where Γ_0 is the "outside" boundary circle. Suppose also that all the boundary circles are centered on the real axis $\mathbb{R}$. Let $t_1, \ldots, t_{N-1}$ be points of $D \cap \mathbb{R}$, such that between each two consecutive 'inside" circles there lies exactly one of the t_j's. Then there is a nonzero analytic differential $f dz$ on $\bar{D}$ such that*

$$\begin{aligned} f dz &\leqslant 0 && \text{on } \Gamma_0, \\ f dz &\geqslant 0 && \text{on } \Gamma_j,\ 1 \leqslant j \leqslant N, \\ f(t_j) &= 0, && 1 \leqslant j \leqslant N - 1. \end{aligned}$$

Proof. We map D conformally onto a slit domain V obtained from the complex plane by excising slits $(-\infty, 0]$, $[a_1, b_1], \ldots, [a_N, b_N]$ along the real axis, so that Γ_0 corresponds to the slit $(-\infty, 0]$. Let w_j be the image of t_j. Then between each pair of consecutive bounded slits there lies exactly one of the w_j's.

Define an analytic differential ω on V by

$$\omega = \frac{(w - w_1) \cdots (w - w_{N-1})\, dw}{i[w(w - a_1)(w - b_1) \cdots (w - a_N)(w - b_N)]^{1/2}},$$

where the branch of the square root is chosen to be positive for large positive values of w (cf. [13, p. 293]). One checks that $\omega \geqslant 0$ along the respective sides of the bounded slits, while $\omega \leqslant 0$ along the sides of the slit $(-\infty, 0]$. The pullback $f dz$ of ω to D has the properties asserted by the lemma.

Now let D and the t_j's be as above. Define $\varphi_0 = -1$ on Γ_0 and $\varphi_0 = 1$ on $\Gamma \backslash \Gamma_0$. If $f dz$ is the differential of the lemma, then $\varphi_0 f dz \geqslant 0$ along Γ. Fix an integer m satisfying

$$0 \leqslant m \leqslant N,$$

and define

$$\varphi_m(z) = \varphi_0 \prod_{j=1}^{m} \frac{z - t_j}{|z - t_j|}, \qquad z \in \Gamma. \tag{6.1}$$

Then

$$\operatorname{ind}(\varphi_m) = m.$$

On the other hand, the analytic function

$$g(z) = f(z) \Big/ \prod_{j=1}^{m} (z - t_j)$$

satisfies $\varphi_m g dz \geqslant 0$, so that φ_m is badly approximable. This shows again that the estimate of Theorem 1.3 cannot be improved upon.

Now consider an infinitely connected domain W obtained from the open unit disc Δ by excising the origin $\{0\}$ together with a sequence of disjoint closed subdiscs $\{\Delta_j\}_{j=1}^{\infty}$, whose centers $\{c_j\}$ lie on the positive real axis and decrease to 0. We claim that for each integer m, there is a badly approximable function φ_m on ∂W with index m. Indeed, let $\{t_j\}_{j=1}^{\infty}$ be a sequence in $W \cap \mathbb{R}$ such that t_j lies between Δ_j and Δ_{j+1}. Define φ_0 to be -1 on $\partial\Delta$ and $+1$ on $(\partial W)\backslash(\partial\Delta)$, and define φ_m as in (6.1). If $D_N = \Delta \backslash \bigcup_{j=1}^{N} \Delta_j$, then the preceding work shows that

$$d(\varphi_m|_{\partial D_N}, A(D_N)) = 1, N \geqslant m.$$

By [5, p. 52], $\bigcup A(D_N)$ is a dense subspace of $A(W)$. Consequently the distances $d(h|_{\partial D_N}, A(D_N))$ decrease to $d(h, A(W))$ whenever $h \in C(\partial W)$. In particular, $d(\varphi_m, A(W)) = \| \varphi_m \|$, so that φ_m is a badly approximable function with index m.

The final example is that of a locally constant unimodular function φ whose range lies on no arc of $\partial\Delta$ of length less than π, but such that φ is not badly approximable. For this we take D to be a circle domain as in Lemma 6.1 with only three boundary circles ($N = 2$), such that D is symmetric with respect to the imaginary axis. In other words, Γ_0 is centered at 0, Γ_1 and Γ_2 have equal radii, and the center of Γ_1 is the negative of the center of Γ_2. Set $\varphi = 1$ on Γ_0, $\varphi = i$ on Γ_1, and $\varphi = -i$ on Γ_2. We claim that $d(\varphi, A(D)) < 1$.

Indeed, suppose that φ is badly approximable. Let $f(z)\,dz$ be a dual extremal differential for φ. Then f is not identically zero, and $\varphi f dz \geqslant 0$. Since $\overline{\varphi(-\bar{z})} = \varphi(z)$, also $\varphi(z)\overline{f(-\bar{z})}\,dz \geqslant 0$. Furthermore the inequality $\varphi f dz \leqslant \varphi[f(z) + \overline{f(-\bar{z})}]\,dz$ shows that $f(z) + \overline{f(-\bar{z})}$ is not identically zero. Replacing f by $f(z) + \overline{f(-\bar{z})}$, we can assume that

$$f(z) = \overline{f(-\bar{z})}, \qquad z \in \bar{D}. \tag{6.2}$$

Let z_0 be the zero of f. Since f has only a single zero, (6.2) shows that $z_0 = -\bar{z}_0$, and thus z_0 lies on the imaginary axis.

According to Lemma 6.1, there is a nonzero analytic differential $g(z)\,dz$ on $\bar{D}$ such that $g(z)\,dz \leqslant 0$ along Γ_0, $g(z)\,dz \geqslant 0$ along $\Gamma_1 \cup \Gamma_2$, and $g(0) = 0$. Set $h = f/g$. If $z_0 = 0$ then h is a bounded analytic function whose argument assumes distinct constant values on the components of Γ, an absurdity. We conclude that $z_0 \neq 0$. Consequently h is meromorphic, h has a simple pole at 0, and h has a simple zero at z_0 (a double zero, if $z_0 \in \Gamma$). Moreover, h maps D conformally onto a slit domain W on the Riemann sphere. If S_0, S_1, and S_2 are the slits that correspond respectively to Γ_0, Γ_1, and Γ_2, then $S_0 \subseteq (-\infty, 0]$, $S_1 \subseteq (i0, i\infty)$, and $S_2 \subseteq (-i0, -i\infty)$. Now replacing g by $g(z) + \overline{g(-\bar{z})}$, we can also assume that g satisfies the same functional relation (6.2) as f. Then also $h(x) = \overline{h(-\bar{z})}$. In other words, the reflection $z \to -\bar{z}$ of D in the imaginary axis corresponds via h to the reflection $w \to \bar{w}$ of the slit domain W.

Now let ψ be the conformal self-map of W which is induced by the conformal map $z \to -z$ of D, that is, $\psi(w) = h(-h^{-1}(w))$. Then ψ leaves the real axis invariant, ψ interchanges the upper and lower half-planes, $\psi(\infty) = \infty$, and $\psi(0) \neq 0$. The map $w \to \overline{\psi(w)}$ then yields an anticonformal self-map $\bar{\psi}$ of the slit upper half-plane $H_+ \backslash S_1$. Now $H_+ \backslash S_1$ is conformally an annulus, and any anticonformal self-map of an annulus is a reflection, which is completely determined by specifying a fixed-point on the boundary. We conclude that $\bar{\psi}$ must be the anticonformal map $w \to -\bar{w}$ of $H_+ \backslash S_1$ because both leave ∞ fixed. However $w \to -\bar{w}$ leaves 0 fixed, whereas $\bar{\psi}$ does not. This contradiction establishes the assertion.

We remark that in the case of a circle domain D with three boundary circles, there is a close relation between the family of locally constant unimodular functions which are badly approximable and the family of conformal maps of D onto radial slit domains on the Riemann sphere. It turns out that the precise description of the badly approximable locally constant functions depends on the size and configuration of the boundary circles of D.

7. Toeplitz Operators

In this section, we indicate the connection between the dual extremal problems under consideration and certain Toeplitz operators. This will lead to another simple proof of Poreda's theorem, and an alternative proof of Theorem 1.3. For details on Toeplitz operators, see [4], which is the source for some of the proofs in this section.

Let τ be a positive finite measure on Γ, and let M be a closed subspace of $L^2(\tau)$ such that

$$A(D)\, M \subset M. \tag{7.1}$$

For fixed $z_0 \in D$, the operator $f \to (z - z_0) f$, $f \in M$, has closed range and null space $\{0\}$. By the theory of Fredholm operators, the range $(z - z_0)\, M$ of these operators has the same codimension (finite or infinite) in M for all $z_0 \in D$. We will be interested in the case that

$$(z - z_0)\, M \text{ has codimension one in } M, \qquad \text{for } z_0 \in D. \tag{7.2}$$

Let P be the orthogonal projection of $L^2(\tau)$ onto M. For each $\varphi \in L^\infty(\tau)$, the Toeplitz operator T_φ is defined on M by

$$T_\varphi f = P(\varphi f), \qquad f \in M.$$

The correspondence $\varphi \to T_\varphi$ is a contractive linear mapping from $L^\infty(\tau)$ to the bounded operators on M, which satisfies $T_\varphi{}^* = T_{\bar\varphi}$ and $T_1 = I$. If $\varphi \in L^\infty(\tau)$ and $\psi \in A(D)$, $T_{\varphi\psi} = T_\varphi T_\psi$ and $T_{\varphi\bar\psi} = T_{\bar\psi} T_\varphi$.

Lemma 7.1. *Suppose that* (7.1) *and* (7.2) *are valid. Then* $T_\varphi T_\psi - T_{\varphi\psi}$ *is a compact operator whenever* $\varphi, \psi \in C(\Gamma)$. *Furthermore, if* $\varphi \in C(\Gamma)$ *does not vanish on* Γ, *then* T_φ *is a Fredholm operator, and*

$$\operatorname{ind}(\varphi) = -\operatorname{index}(T_\varphi). \tag{7.3}$$

Proof. Here

$$\operatorname{index}(T_\varphi) = \dim \mathcal{N}(T_\varphi) - \operatorname{cod} \mathcal{R}(T_\varphi)$$

where $\mathscr{R}$ denotes "range" and $\mathscr{N}$ denotes "null space." Now $T_\varphi T_\psi - T_{\varphi\psi} = 0$ when $\psi \in A(D)$. If $\psi(z) = 1/(z - z_0)$ for some fixed $z_0 \in D$, then $T_\varphi T_\psi = T_{\varphi\psi}$ on $(z - z_0) M$, so that $T_\varphi T_\psi - T_{\varphi\psi}$ is one-dimensional, hence compact. Since linear combinations of functions in $A(D)$ and the functions $1/(z - z_0)$, $z_0 \in D$, are dense in $C(\Gamma)$ [3], $T_\varphi T_\psi - T_{\varphi\psi}$ is compact for all $\varphi, \psi \in C(\Gamma)$.

Suppose $\varphi \in C(\Gamma)$ does not vanish anywhere on Γ. For $z_0 \in D$ fixed, we can express

$$\varphi(z) = (z - z_0)^m gh,$$

where $m = \operatorname{ind}(\varphi)$, g is an invertible function in $A(D)$, and $h \in C(\Gamma)$ has a continuous logarithm on Γ. Then h is appropriately homotopic to the constant function 1, so that T_h has index zero. Since T_g is invertible, its index is zero. By (7.2), the index of the Toeplitz operator of $(z - z_0)^m$ is $-m$. Consequently index $(T_\varphi) = -m = -\operatorname{ind}(\varphi)$. Q.E.D.

The usual Toeplitz operators are obtained by setting D equal to the open unit disc Δ, and setting $M = H^2(d\theta)$. In this case, Poreda's theorem can be proved as follows. Assume $\varphi \in C(\partial D)$ is unimodular. Then $d(\varphi, A(\Delta)) < 1$ if and only if T_φ is left invertible, that is, if and only if $\dim \mathscr{N}(T_\varphi) > 0$ [4, p. 187]. If then φ is badly approximable, we have $\dim \mathscr{N}(T_\varphi) > 0$. By Coburn's lemma [4, p. 185], $\operatorname{cod} \mathscr{R}(T_\varphi) = 0$ so (7.3) shows that $\operatorname{ind}(\varphi) < 0$. On the other hand, if φ is not badly approximable, then $\dim \mathscr{N}(T_\varphi) = 0$ and (7.3) yields $\operatorname{ind}(\varphi) \geqslant 0$, which does it.

To extend this proof, we require an analog of Coburn's lemma, and a criterion relating the distance estimate to left invertibility. A criterion sufficient for our purposes can be found in the work of Abrahamse [1]. The precise fact we will need can be proved for infinitely connected domains. It is the following.

LEMMA 7.2. *Let $\varphi \in C(\Gamma)$ be unimodular. Then φ is badly approximable if and only if there are a positive measure τ on Γ and a subspace M of $L^2(\tau)$ satisfying* (7.1) *and* (7.2), *such that the Toeplitz operator T_φ on M is not left invertible, that is, $\mathscr{N}(T_\varphi) \neq \{0\}$.*

Proof. If φ is not badly approximable, there is $g \in A(D)$ satisfying $\| g - \varphi \| < 1$. Since φ is unimodular, $\| 1 - g\bar{\varphi} \| < 1$. Hence $\| T_{1-\bar{g}\varphi} \| = \| I - T_{\bar{g}} T_\varphi \| < 1$, so that $T_{\bar{g}} T_\varphi$ is invertible, and T_φ is left invertible.

On the other hand, suppose that φ is badly approximable. Let $\mu \in A(D)^\perp$ be a dual extremal measure for φ, so that $\varphi\mu \geqslant 0$. Let $\tau = \varphi\mu$, and let $M = H^2(\tau)$ be the closure of $A(D)$ in $L^2(\tau)$. If $g \in A(D)$, then $\int g\bar{\varphi}\, d\tau = \int g\, d\mu = 0$, so that $\varphi \perp H^2(\tau)$. From the definition of T_φ, we obtain $T_\varphi(1) = 0$, and $1 \in \mathscr{N}(T_\varphi)$. It suffices now to establish (7.2).

Suppose that $(z - z_0) M = M$. Then $1/(z - z_0)^m \in M$ for all integers

$m \geqslant 0$. Hence $\int (1/(z - z_0)^m)\, d\mu = 0$ for all $m \geqslant 1$. Hence μ is orthogonal to the linear span of the functions in $A(D)$ and the $1/(z - z_0)^m$, $m \geqslant 1$ [3]. Since this linear span is dense in $C(\Gamma)$, we obtain $\mu = 0$, a contradiction.

It follows that the closed subspace $(z - z_0)\, M$ of M has codimension at least 1 in M. Since $(z - z_0)\, A(D)$ has codimension 1 in $A(D)$, $(z - z_0)\, M$ has codimension precisely 1 in M, and (7.2) is valid.

The required analog of Coburn's Lemma is as follows.

LEMMA 7.3. *Suppose that Γ consists of $N + 1$ disjoint simple closed analytic Jordan curves. Let τ be a positive measure on Γ which is absolutely continuous with respect to the arc length measure on Γ. If $\varphi \in L^\infty(\tau)$ satisfies $\mathcal{N}(T_\varphi) \neq \{0\}$, then*

$$\dim \mathcal{N}(T_{\bar{\varphi}}) \leqslant N.$$

Proof. Let $f \in \mathcal{N}(T_\varphi)$, $f \neq 0$, Then

$$\int \varphi f \bar{h}\, d\tau = 0, \qquad \text{all } h \in M. \tag{7.4}$$

In particular, $\int \varphi\, |f|^2\, \bar{\psi}\, d\tau = 0$ for all $\psi \in A(D)$. Consequently $\bar{\varphi}\, |f|^2\, d\tau$ is an analytic differential of class H^1. It follows that τ is mutually absolutely continuous with respect to arc length ds, and that f cannot vanish on a set of positive measure.

Now let $g \in \mathcal{N}(T_{\bar{\varphi}})$. Then $\int \bar{\varphi}\, g\bar{h} d\tau = 0$ for all $h \in M$, so that $\int \bar{\varphi}\, g\overline{f\psi} d\tau = 0$ for all $\psi \in A(D)$. Setting $h = \psi g$ in (7.4), we find also that $\int \bar{\varphi}\, g\bar{f}\overline{\psi} d\tau = 0$ for all $\psi \in A(D)$. Hence $\bar{\varphi} g\bar{f} d\tau$ is orthogonal to $A(D) + \overline{A(D)}$. Since this latter space has defect N in $C(\Gamma)$, and since f cannot vanish on a set of positive measure, the collection of such g's has dimension at most N.

Alternative Proof of Theorem 1.3. We can assume that Γ consists of $N + 1$ simple closed analytic Jordan curves. Let $\varphi \in C(\Gamma)$ be a unimodular badly approximable function. We will show that $\operatorname{ind}(\varphi) < N$.

Take τ and M as in Lemma 7.2, so that $\mathcal{N}(T_\varphi) \neq \{0\}$. Note that the τ chosen in Lemma 7.2 is the variation of a measure in $A(D)^\perp$, so that in the case at hand, we can assume that τ is the modulus of an analytic differential, hence absolutely continuous with respect to arc length measure on Γ. By Lemma 7.3, and the relations $T_{\bar{\varphi}} = T_\varphi{}^*$, $\mathcal{N}(T_{\bar{\varphi}}{}^*) = \mathcal{R}(T_\varphi)^\perp$, we obtain

$$\operatorname{cod} \mathcal{R}(T_\varphi) \leqslant N.$$

From Lemma 7.1 we obtain

$$\operatorname{ind}(\varphi) = \operatorname{cod} \mathcal{R}(T_\varphi) - \dim \mathcal{N}(T_\varphi) \leqslant N - 1.$$

This completes the proof.

Note that the estimate of Lemma 7.3 is sharp. Indeed, Section 6 provides a circle domain D bounded by $N + 1$ circles, and a unimodular $\varphi \in C(\Gamma)$ such that φ is badly approximable, while $\operatorname{ind}(\varphi) = N - 1$. Choosing τ and M as in Lemma 7.2, we obtain $\dim \mathcal{N}(T_{\bar{\varphi}}) = \operatorname{cod} \mathcal{R}(T_\varphi) = \operatorname{ind}(\varphi) + \dim \mathcal{N}(T_\varphi) \geqslant N$, so that in fact equality must hold. An example in Section 6 also shows that there are infinitely connected domains for which no estimate as in Lemma 7.3 obtains.

8. Riemann Surfaces

In this section, we indicate how some of the results of this paper can be extended to Riemann surfaces. Let D be a finite bordered Riemann surface with interior genus P, such that the boundary Γ of D consists of $N + 1$ closed analytic curves. Again $A(D)$ is the algebra of analytic functions on D which extend continuously to Γ, and $A(D)^\perp$ consists of measures on Γ which are the boundary values of analytic differentials on D of class H^1. The proof of Theorem 1.1 is valid in this context. The analogue of Theorem 1.3 is the following.

Theorem 8.1. *If $\varphi \in C(\Gamma)$ is badly approximable, then φ has nonzero constant modulus, and*

$$\operatorname{ind}(\varphi) < 2P + N.$$

The theory of Toeplitz operators developed in Section 7 also carries over to this context. Fix a function F analytic on $\bar{D}$ such that F has only one zero on $\bar{D}$, a simple zero at some point of D. Let τ be a finite measure on Γ, and let M be a closed subspace of $L^2(\tau)$ such that

$$A(D)\, M \subseteq M, \tag{8.1}$$

$$FM \text{ has codimension one in } M. \tag{8.2}$$

The Toeplitz operators T_φ on M are defined as before, and Lemma 7.1 is valid. The proof of Lemma 7.1 also carries over to this context, once one makes the following two observations: First, the linear span of $A(D)$ and the functions $1/F^m$, $m \geqslant 1$, is dense in $C(\Gamma)$ [11]. Secondly, if φ is a nonvanishing function on Γ with index m, then there are $h \in C_R(\Gamma)$ and an invertible function $g \in A(D)$ such that $\varphi = F^m g \exp(h)$.

The proof of Lemma 7.2 also carries over, once one replaces $z - z_0$ by F. Lemma 7.3 is also valid, except that one obtains only

$$\mathcal{N}(T_\varphi) \neq \{0\} \text{ implies } \dim \mathcal{N}(T_{\bar{\varphi}}) \leqslant 2P + N, \tag{8.3}$$

because $A(D) + \overline{A(D)}$ has defect $2P + N$ in $C(\Gamma)$ [11]. The alternative proof of Theorem 1.3 given in Section 7 then serves to establish the estimate given in Theorem 8.1.

Again the estimates of Theorem 8.1 and the analog (8.3) of Coburn's lemma are sharp. To see this, we proceed as follows.

Let α be any analytic differential on $\bar{D}$ which has no zeros, and let τ be the measure on Γ defined by $\tau = |\alpha|$. Then $\tau = \varphi\alpha$, where φ is a continuous unimodular function on Γ. Furthermore, α is a dual extremal differential for φ, so that φ is badly approximable. Let M be the closure of $A(D)$ in $L^2(\tau)$, and consider the Toeplitz operator T_φ on M. Since

$$0 = \int g\alpha = \int g\bar{\varphi}\, d\tau, \qquad \text{all } g \in A(D),$$

the projection of φ into M is 0, and

$$T_\varphi(1) = 0.$$

Let ω be a Schottky differential for D, that is, ω is an analytic differential on $\bar{D}$ which is real along Γ (cf. [11]). Then $\omega/\alpha = h$ is analytic on $\bar{D}$. Moreover, if $g \in A(D)$, then

$$0 = \int_\Gamma \bar{g}\omega = \int_\Gamma \bar{g}h\alpha = \int \bar{\varphi}h\bar{g}\, d\tau.$$

It follows that $h \in \mathcal{N}(T_{\bar{\varphi}})$. Since the dimension of the space of Schottky differentials is $2P + N$, the dimension of $\mathcal{N}(T_{\bar{\varphi}})$ is at least $2P + N$, so that from (8.3) its dimension is precisely $2P + N$, and in particular the estimate (8.3) is sharp. One checks that only the constants lie in $\mathcal{N}(T_\varphi)$, so that $\dim \mathcal{N}(T_\varphi) = 1$, and

$$\operatorname{ind}(\varphi) = -\operatorname{index}(T_\varphi) = 2P + N - 1.$$

Hence the estimate of Theorem 8.1 is also sharp.

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